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The three primes theorem with almost equal summands

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The problem of representing all sufficiently large odd numbers as the sum of three nearly equal primes is tackled using the Hardy–Littlewood circle method in tandem with a sieve method. A combination of the Harman and vector sieves, as developed by Baker, Harman and Pintz, is used. To do this, the major arcs of the circle method involve the investigation of the mean-values of Dirichlet polynomials, while the minor arcs demand short-range estimates for exponential sums.

Keywords: Hardy–Littlewood circle method; sieve methods; Goldbach–Vinogradov theorem; exponential sums; Dirichlet polynomials; distribution of primes

1. Introduction

Every sufficiently large odd integer N is a sum of three primes (Vinogradov 1937; see also Davenport 1980). Haselgrove (1951) showed that each prime summand may be taken from the interval

$$[\tfrac{1}{3}N - N^{\theta}, \tfrac{1}{3}N + N^{\theta}] \tag{1.1}$$

provided that $\frac{63}{64} < \theta < 1$. It is natural to attempt to extend the range of θ (for papers on this topic, see Cheng-Dong 1959; Jingrun 1965; Cheng-Dong & Cheng-Biao 1989; Chaohua 1989, 1991*a*–*e*, 1994; Zhan 1991) Recently Chaohua (1994) and H. Mikawa (1994, unpublished work) have given this result for

$$\tfrac{7}{12} < \theta < 1.$$

In the present paper we sharpen this as follows.

Theorem 1.1. *Suppose that* $\frac{4}{7} \leqslant \theta < 1$. *Every sufficiently large odd integer* N *is the sum of three primes from the interval (1.1).*

With more work it may be possible to obtain the exponent $\frac{9}{16}$, but the value $\frac{11}{20}$ seems out of reach without a substantial new idea.

The method overlaps with that of Baker *et al*. (1997); from which we shall quote a number of results. In that paper the following simple observation is crucial. Let I, J be intervals. Let ρ denote the indicator function of the prime numbers and suppose that

$$A_0(k) \leqslant \rho(k) \leqslant A_1(k) \quad (k \in I), \tag{1.2}$$

$$B_0(m) \leqslant \rho(m) \leqslant B_1(m) \quad (m \in J) \tag{1.3}$$

for certain real sequences $A_j(k), B_j(m)$. Then

$$\rho(k)\rho(m) \geqslant A_0(k)B_1(m) + A_1(k)B_0(m) - A_1(k)B_1(m) \tag{1.4}$$

for $(k,m) \in I \times J$. In the present paper we are concerned with

$$I = J = [\tfrac{1}{3}N - Y, \tfrac{1}{3}N], \quad Y = N^\theta, \quad N > C_1(\theta)$$

for a given $\theta \in [\frac{4}{7}, 1)$. We find large classes of sequences $\boldsymbol{b} = (b(k))_{k\in I}$, $\boldsymbol{c} = (c(m))_{m\in I}$ such that

$$\sum_{\substack{k,m\in I\\ k+m=2n}} b(k)c(m) = \frac{u(\boldsymbol{b})u(\boldsymbol{c})}{\mathcal{L}^2}(\tfrac{2}{3}N - 2n)\mathfrak{S}(2n)(1 + O(\mathcal{L}^{-1})) \tag{1.5}$$

for almost all even integers $2n$ in $K = [\frac{2}{3}N - Y, \frac{2}{3}N - \frac{1}{2}Y]$; the number of exceptional $2n$ is $O(Y\mathcal{L}^{-2})$. Here $\mathcal{L} = \log(N/3)$; $\mathfrak{S}(2n)$ is the singular series associated with Goldbach's problem; and $u(\boldsymbol{b}), u(\boldsymbol{c})$ are 'density' constants (see theorem 1.2).

Each pair $(\boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{A}_i, \boldsymbol{B}_j)$ will satisfy (1.5), and moreover, we shall find that

$$u(\boldsymbol{A}_0)u(\boldsymbol{B}_1) + u(\boldsymbol{A}_1)u(\boldsymbol{B}_0) > u(\boldsymbol{A}_1)u(\boldsymbol{B}_1). \tag{1.6}$$

It follows from (1.4)–(1.6) that all but $O(Y\mathcal{L}^{-2})$ even integers $2n$ in K may be written in the form

$$2n = p_1 + p_2 \quad (p_i \in I).$$

(The letter p is reserved for a prime variable.) Moreover, the number of even integers in K of the form $N - p_3$ with $p_3 \in [\frac{1}{3}N + \frac{1}{2}Y, \frac{1}{3}N + Y]$ is $\gg Y\mathcal{L}^{-1}$ (Heath-Brown & Iwaniec 1979; see Baker *et al*. 1997). Thus there are $\gg Y\mathcal{L}^{-1}$ primes $p_3 \in [\frac{1}{3}N + \frac{1}{2}Y, \frac{1}{3}N + Y]$ for which

$$N - p_3 = p_1 + p_2, \quad (p_1, p_2) \in I \times I,$$

which yields theorem 1.1.

Before going further we specify the classes $\mathcal{B}_0, \mathcal{B}$ and $\mathcal{C}$ of real sequences, defined on the integers in I, for which we shall prove (1.5) when $\boldsymbol{b} \in \mathcal{B}_0 \cap \mathcal{B}$ and either $\boldsymbol{b} \in \mathcal{C}, \boldsymbol{c} \in \mathcal{B}_0$ or $\boldsymbol{c} \in \mathcal{B}_0 \cap \mathcal{C}$. Implied constants may depend on A and ϵ; B denotes an absolute constant, which need not be the same at each occurrence.

We write $\boldsymbol{b} \in \mathcal{B}_0$ if, for every $A > 0$:

(i) we have

$$\sum_{k\in I, k\leqslant t}\left(b(k)\chi(k) - \frac{\delta_\chi u}{\mathcal{L}}\right) \ll Y\mathcal{L}^{-A} \tag{1.7}$$

for a constant $u = u(\boldsymbol{b})$ and for any real t and Dirichlet character $\chi \pmod q$, $q \leqslant \mathcal{L}^A$;

(ii) we have $b(k) = 0$ unless

$$(k, P(\mathcal{L}^A)) = 1, \tag{1.8}$$

where

$$P(z) = \prod_{p<z} p;$$

(iii) we have

$$b(k) = O(\tau(k)^B). \tag{1.9}$$

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We say that $\boldsymbol{b} \in \mathcal{B}$ if $\boldsymbol{b}$ has properties (ii), (iii) and if, in addition,

(iv) we have

$$\sum_{k \in I} |b(k) - b'(k)| \ll Y\mathcal{L}^{-A} \tag{1.10}$$

for a real sequence $b'(k)$ with the following property:

$$\int_{T'}^{T'+T} \left| \sum_{k \leqslant N} \frac{b'(k)\chi(k)}{k^{1/2+it}} \right| \mathrm{d}t \ll TYN^{-1/2}\mathcal{L}^{-A} \tag{1.11}$$

for any character $\chi \pmod q$, whenever

$$q \leqslant \mathcal{L}^A, \quad T \in [NY^{-1}, N], \quad T_0 \leqslant T' \ll T^2, \quad T' + T \leqslant N. \tag{1.12}$$

Here $T_0 = \exp(\mathcal{L}^{1/3})$.

Let ϵ be a sufficiently small positive constant depending on θ. We write $\boldsymbol{b} \in \mathcal{C}$ if $\boldsymbol{b}$ has properties (ii), (iii) and $\boldsymbol{b}$ satisfies (1.10), where $\boldsymbol{b}'$ is the sum of the following sequences:

(I) a sequence

$$c'(k) = \sum_{\substack{st=k \\ s \ll N^{1/2}}} f(s), \quad f(s) \ll \mathcal{L}^B;$$

(II) at most $\mathcal{L}$ sequences of the form

$$c''(k) = \sideset{}{^*}\sum_{\substack{m_1 \ldots m_r = k \\ M_i < m_i \leqslant 2M_i}} f_1(m_1) \ldots f_r(m_r),$$

where some subproduct of the M_i lies in $[N^{1-\theta+\epsilon}, N^{\theta-\epsilon}]$ and $(*)$ indicates $O(1)$ relations

$$m_1^{e_1} \ldots m_r^{e_r} \geqslant X$$

with absolute constants $e_1, \ldots, e_r$, and where $1 \leqslant X \leqslant N$. Moreover,

$$f_1(m_1) \ldots f_r(m_r) \ll \tau(m_1 \ldots m_r)^B.$$

Theorem 1.2. *Let $\theta \in (\frac{1}{2}, 1)$ and $N > C_2(A, \theta)$. Let $\boldsymbol{b} \in \mathcal{B}_0 \cap \mathcal{B}$, $\boldsymbol{c} \in \mathcal{B}_0$. Suppose one of $\boldsymbol{b}$ or $\boldsymbol{c}$ is in $\mathcal{C}$. Then (1.5) holds for all but at most $Y\mathcal{L}^{-A}$ even integers in K.*

The proof of theorem 1.2 will be given in § 2. In § 3 we shall develop families of sequences that belong to $\mathcal{B} \cap \mathcal{C}$, culminating in pairs $(\boldsymbol{A}_0, \boldsymbol{B}_1)$, $(\boldsymbol{A}_1, \boldsymbol{B}_0)$, $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ in $(\mathcal{B}_0 \cap \mathcal{B} \cap \mathcal{C}) \times \mathcal{B}_0$ for which (1.4) and (1.6) hold.

2. Proof of theorem 1.2

We may suppose that A is large. Let $Q = [Y^2 N^{-1} \mathcal{L}^{-2A}]$,

$$I_{q,r} = \left[\frac{r}{q} - \frac{1}{qQ}, \frac{r}{q} + \frac{1}{qQ} \right]$$

for $1 \leqslant q \leqslant Q$, $1 \leqslant r \leqslant q$, $(r, q) = 1$. We write

$$\mathcal{M} = \bigcup_{q \leqslant \mathcal{L}^{2A}} \sideset{}{^*}\bigcup_{r=1}^{q} I_{q,r};$$

in this section an asterisk denotes the condition $(r, q) = 1$. Further, let

$$\mathcal{M}^c = [1/Q, 1+1/Q] \backslash \mathcal{M}.$$

The left-hand side of (1.5) is

$$\int_{1/Q}^{1+1/Q} f(\alpha)g(\alpha)e(-2n\alpha)\,\mathrm{d}\alpha,$$

where

$$e(\theta) = \mathrm{e}^{2\pi\mathrm{i}\theta}, f(\alpha) = \sum_{k\in I} b(k)e(k\alpha), g(\alpha) = \sum_{m\in I} c(m)e(m\alpha).$$

In order to prove theorem 1.2, it suffices to show that

$$\sum_{2n\in K}\left|\int_{\mathcal{M}} f(\alpha)g(\alpha)e(-2n\alpha)\,\mathrm{d}\alpha - \frac{u(\boldsymbol{b})u(\boldsymbol{c})(\frac{2}{3}N-2n)\mathfrak{S}(2n)}{\mathcal{L}^2}\right|^2 \ll Y^3\mathcal{L}^{-A-7}, \quad (2.1)$$

and that

$$\sum_{2n\in K}\left|\int_{\mathcal{M}^c} f(\alpha)g(\alpha)e(-2n\alpha)\,\mathrm{d}\alpha\right|^2 \ll Y^3\mathcal{L}^{-A-7}. \quad (2.2)$$

We deal with (2.2) rather quickly. Suppose for example that $\boldsymbol{b} \in \mathcal{C}$. Let $\Gamma(\alpha)$ be the indicator function of $\mathcal{M}^c$. By Parseval's equality and (1.8), the left-hand side of (2.2) is

$$\begin{aligned}&\leqslant \int_0^1 |\Gamma(\alpha)f(\alpha)g(\alpha)|^2\,\mathrm{d}\alpha\\ &\leqslant \max_{\alpha\in\mathcal{M}^c}|f(\alpha)|^2\int_0^1|g(\alpha)|^2\,\mathrm{d}\alpha \ll Y\mathcal{L}^B \max_{\alpha\in\mathcal{M}^c}|f(\alpha)|^2.\end{aligned}$$

Since there is $\boldsymbol{b}'$, satisfying (1.10), that decomposes into type (I) and (II) sums as explained above, it suffices to prove that, for $\alpha \in \mathcal{M}^c$,

$$\begin{aligned}S_1(\alpha) :&= \sum_{s\leqslant N^{1/2}}\left|\sum_{st\in I} e(st\alpha)\right| \ll Y\mathcal{L}^{-A},\\ S_2(\alpha) :&= \sum_{\substack{M<s\leqslant 2M\\ st\in I}} \beta(s)\gamma(t)e(st\alpha) \ll Y\mathcal{L}^{-A+B} \quad (2.3)\end{aligned}$$

for $M \in [N^{1-\theta+\epsilon}, N^{1/2}]$; here $|\beta(s)\gamma(t)| \ll \tau(st)^B$. (The relations $m_1^{e_1}\dots m_r^{e_r} \geqslant X$ permitted in the definition of $\boldsymbol{c}''$ may be removed by a standard application of Perron's formula.)

Since $\alpha \in \mathcal{M}^c$, Dirichlet's theorem yields a rational approximation to α of the form

$$\left|\alpha - \frac{r}{q}\right| < \frac{1}{qQ}, \quad \mathcal{L}^{2A} < q \leqslant Q, \quad (r, q) = 1.$$

By (3) of Davenport (1980, § 25),

$$S_1(\alpha) \ll \mathcal{L}^B\left(\frac{Y}{q} + N^{1/2} + q\right) \ll Y\mathcal{L}^{-A}.$$

Applying Cauchy's inequality to $S_2(\alpha)$,

$$|S_2(\alpha)|^2 \ll M\mathcal{L}^B \sum_{M<s\leqslant 2M} \Bigg| \sum_{\substack{t \\ st\in I}} \gamma(t)e(st\alpha)\Bigg|^2$$

$$\ll M\mathcal{L}^B \sum_{t_1\in L, t_2\in L} |\gamma(t_1)\gamma(t_2)| \Bigg|\sum_s e(s(t_1-t_2)\alpha)\Bigg|,$$

where $L=[\frac{1}{7}NM, \frac{1}{2}NM]$ and s is summed over the interval

$$M < s \leqslant 2M, \quad st_1 \in I, \quad st_2 \in I.$$

Since $|\gamma(t_1)\gamma(t_2)| \leqslant \frac{1}{2}(|\gamma(t_1)|^2 + |\gamma(t_2)|^2)$, we have

$$|S_2(\alpha)|^2 \ll M\mathcal{L}^B \sum_{t_1\in L} |\gamma(t_1)|^2 \sum_{\substack{t_2 \\ |t_2-t_1|\leqslant YM^{-1}}} \min\left(\frac{YM}{N}, \frac{1}{||(t_1-t_2)\alpha||}\right)$$

$$\ll M\mathcal{L}^B \frac{N}{M}\frac{YM}{N}$$

$$+ M\mathcal{L}^B \sum_{t_1\in L} |\gamma(t_1)|^2 \sum_{0<|t_2-t_1|\leqslant YM^{-1}} \min\left(\frac{YM}{N}, \frac{1}{||(t_1-t_2)\alpha||}\right)$$

$$\ll YM\mathcal{L}^B + M\mathcal{L}^B \frac{N}{M}\left(\frac{Y^2}{Nq} + \frac{Y}{M} + Q\right),$$

on a further application of Davenport (1980, § 25, (3)). Since $\mathcal{L}^{2A} < q \leqslant Q$, (2.3) follows from our choice of Q above.

Turning to a point α in $\mathcal{M}$, say $\alpha = r/q+\eta \in I_{q,r}$, we shall give the approximation

$$f(\alpha) = f_0(\alpha) + O(Y\mathcal{L}^{-A}), \tag{2.4}$$

where

$$f_0(\alpha) = \frac{\mu(q)}{\phi(q)}\frac{u(\boldsymbol{b})}{\mathcal{L}}t(\eta), \quad t(\eta) = \sum_{k\in I} e(k\eta).$$

By (2.4) of Baker *et al*. (1997), it suffices to show, for a fixed $\chi \pmod q$, that

$$S_1(\chi,\eta) := \sum_{k\in I} b(k)\chi(k)e(k\eta) = \frac{u(\boldsymbol{b})\delta_\chi}{\mathcal{L}}\ t(\eta) + O(Y\mathcal{L}^{-2A}). \tag{2.5}$$

We rewrite $S_1(\chi,\eta)$ in the form

$$S_1(\chi,\eta) = \int_I e(v\eta)\,\mathrm{d}G_1(v) + O(Y\mathcal{L}^{-2A}), \tag{2.6}$$

where

$$G_1(v) = \sum_{w\leqslant k<v} b'(k)\chi(k), \quad w = \tfrac{1}{3}N - Y.$$

We now apply Perron's formula (Titchmarsh 1986, lemma 3.12) to obtain

$$G_1(v) = G_2(v) + O(1) \quad (v \in I),$$

where

$$G_2(v) = \frac{1}{2\pi \mathrm{i}} \int_{(1/2)-\mathrm{i}N}^{(1/2)+\mathrm{i}N} \sum_{k \leqslant N} b'(k)\chi(k)k^{-s} \left(\frac{v^s - w^s}{s} \right) \mathrm{d}s.$$

Consequently,

$$\begin{aligned} &\int_I e(v\eta)\,\mathrm{d}G_1(v) - \int_I e(v\eta)\,\mathrm{d}G_2(v) \\ &\quad = [e(v\eta)(G_1(v) - G_2(v))]_w^{N/3} - 2\pi \mathrm{i}\eta \int_I e(v\eta)(G_1(v) - G_2(v))\,\mathrm{d}v \\ &\quad \ll (1 + Y|\eta|) \max_{v \in I} |G_1(v) - G_2(v)| \ll Y\mathcal{L}^{-2A}. \end{aligned} \tag{2.7}$$

Accordingly we prove (2.5) with

$$S_2(\chi, \eta) := \int_I e(v\eta)\,\mathrm{d}G_2(v) \tag{2.8}$$

in place of $S_1(\chi, \eta)$.

We observe that

$$\begin{aligned} S_2(\chi, \eta) &= \frac{1}{2\pi} \int_I e(v\eta) \int_{-N}^{N} F(t, \chi) v^{-1/2+\mathrm{i}t}\,\mathrm{d}t\mathrm{d}v \\ &= S_3(\chi, \eta) + J, \end{aligned} \tag{2.9}$$

say, where

$$\left. \begin{aligned} F(t, \chi) &= \sum_{k \leqslant N} \frac{b'(k)\chi(k)}{k^{1/2+\mathrm{i}t}}, \\ S_3(\chi, \eta) &= \frac{1}{2\pi} \int_I e(v\eta) \int_{-T_0}^{T_0} F(t, \chi) v^{-1/2+\mathrm{i}t}\,\mathrm{d}t\mathrm{d}v, \\ J = J(\chi, \eta) &= \frac{1}{2\pi} \int_{|t| \in [T_0, N]} F(t, \chi) \int_I e(v\eta) v^{-1/2+\mathrm{i}t}\,\mathrm{d}v\mathrm{d}t. \end{aligned} \right\} \tag{2.10}$$

To show that J is an error term, we first appeal to lemmata 4.3 and 4.5 of Titchmarsh (1986) to obtain

$$J \ll \int_{|t| \in [T_0, N]} |F(t, \chi)| \min \left(YN^{-1/2}, N^{1/2}|t|^{-1/2}, N^{-1/2} \min_{v \in I} \left| \frac{t}{v} + 2\pi\eta \right| \right) \mathrm{d}t. \tag{2.11}$$

We are now in a position to apply (1.11). Let

$$\begin{aligned} L_1 &= \{t : |t| \in [T_0, NY^{-1}]\}, \\ L_2 &= \{t : |t| \in [\pi|\eta|N, N], |t| > NY^{-1}\} \\ L_3 &= \{t : NY^{-1} < |t| \leqslant \pi|\eta|N\}. \end{aligned}$$

The contribution to J from L_1 is

$$\begin{aligned} &\ll YN^{-1/2} \int_{[T_0, NY^{-1}]} (|F(t, \chi)| + |F(t, \bar{\chi})|)\,\mathrm{d}t \\ &\ll YN^{-1/2} NY^{-1} YN^{-1/2} \mathcal{L}^{-3A} \ll Y\mathcal{L}^{-3A} \end{aligned} \tag{2.12}$$

from (1.11) with $T' = T_0$, $T = NY^{-1}$.

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For $(t,v)\in L_2\times I$, we have

$$\left|\frac{t}{v}+2\pi\eta\right|\geqslant\frac{3|t|}{N}-2\pi|\eta|>\frac{|t|}{N}.$$

The contribution to J from L_2 is thus

$$\begin{aligned}&\ll N^{1/2}\mathcal{L}\sup_{NY^{-1}\leqslant T\leqslant N/2}T^{-1}\int_T^{2T}(|F(t,\chi)+|F(t,\bar\chi)|)\,\mathrm{d}t\\&\ll Y\mathcal{L}^{-3A}\end{aligned}\tag{2.13}$$

from (1.11) with $T'=T\in[NY^{-1},\frac{1}{2}N]$.

In considering L_3 we may suppose that

$$NY^{-1}<\pi|\eta|N,\tag{2.14}$$

so that $$Z:=\max(NY^{-1},\pi|\eta|Y)<\pi|\eta|N.$$

Now let $$L_3(j)=\{t\in L_3: jZ\leqslant t<(j+1)Z\}$$

so that $$|j|Z\leqslant\pi|\eta|N+Z<2\pi|\eta|N$$

for non-empty $L_3(j)$.

For $t\in L_3(j)$, $v\in I$,

$$\begin{aligned}\left|\frac{t}{v}-\frac{jZ}{\frac{1}{3}N}\right|&=\left|\frac{(t-jZ)\frac{1}{3}N+jZ(\frac{1}{3}N-v)}{v\frac{1}{3}N}\right|\\&\leqslant\frac{Z}{v}+\frac{2\pi|\eta|NY}{v\frac{1}{3}N}\leqslant\frac{7Z}{v}<\frac{22Z}{N}.\end{aligned}\tag{2.15}$$

We now distinguish two cases.

Case 1. We have

$$\left|\frac{jZ}{\frac{1}{3}N}+2\pi\eta\right|\geqslant\frac{44Z}{N}.$$

Then for $t\in L_3(j)$, $v\in I$,

$$\begin{aligned}\left|\frac{t}{v}+2\pi\eta\right|&\geqslant\left|\frac{jZ}{\frac{1}{3}N}+2\pi\eta\right|-\left|\frac{t}{v}-\frac{jZ}{\frac{1}{3}N}\right|\\&\geqslant\frac{1}{2}\left|\frac{jZ}{\frac{1}{3}N}+2\pi\eta\right|\end{aligned}$$

from (2.15).

The contribution to $J(\chi,\eta)$ from $L_3(j)$ is thus

$$\ll N^{1/2}Z^{-1}\left(1+\left|j+\frac{2\pi\eta N}{3Z}\right|\right)^{-1}\int_{L_3(j)}|F(t,\chi)|\,\mathrm{d}t\tag{2.16}$$

$$\ll N^{1/2}Z^{-1}\left(1+\left|j+\frac{2\pi\eta N}{3Z}\right|\right)^{-1}ZYN^{-1/2}\mathcal{L}^{-3A}\tag{2.17}$$

from (1.11) with $NY^{-1}\leqslant T'\leqslant\pi|\eta|N$, $T=Z$.

For the last step, we must verify condition (1.12):

$$|\eta|N=(NY^{-1})(|\eta|Y)\ll Z^2.$$

Case 2. We have

$$\left|\frac{jZ}{\frac{1}{3}N}+2\pi\eta\right|<\frac{44Z}{N}.$$

Let $t\in L_3(j)$. Then

$$2\pi|\eta|N\leqslant|2\pi\eta N+3jZ|+3|j|Z\ll|j|Z\ll|t|$$

for $j\neq 0$, while if $j=0$ we have

$$2\pi|\eta|N<44Z,$$

which implies that $Z=NY^{-1}$ and

$$|t|\geqslant Z\gg|\eta|N.$$

Thus for all j,

$$\begin{aligned}\min(YN^{-1/2},N^{1/2}|t|^{-1/2})&\ll\min(YN^{-1/2},|\eta|^{-1/2})\\&=\frac{\max(N^{1/2},|\eta|^{1/2}Y)}{\max(NY^{-1},|\eta|Y)}\\&\ll N^{1/2}\mathcal{L}^AZ^{-1}\end{aligned}$$

from the definitions of $\mathcal{M}$ and Z. In place of (2.12) we have the bound

$$\ll N^{1/2}\mathcal{L}^AZ^{-1}\int_{L_3(j)}|F(t,\chi)|\,\mathrm{d}t\ll Y\mathcal{L}^{-2A} \tag{2.18}$$

by a similar application of (1.11).

From (2.12), (2.13), (2.17) and (for $O(1)$ values of j) (2.18), we get

$$J\ll Y\mathcal{L}^{-2A}. \tag{2.19}$$

It now suffices to prove (2.5) with $S_3(\chi,\eta)$ in place of $S_1(\chi,\eta)$.

To extract a main term from $S_3(\chi,\eta)$, we verify readily that, for $v\in I$,

$$\begin{aligned}v^{-1/2+\mathrm{i}t}&=w^{-1/2+\mathrm{i}t}+O\left(|-\tfrac{1}{2}+\mathrm{i}t|w^{-1/2}\left|\frac{v}{w}-1\right|\right)\\&=w^{-1/2+\mathrm{i}t}+O(T_0YN^{-3/2}).\end{aligned}$$

An appeal to (1.9) yields

$$S_3(\chi,\eta)=K_\chi\int_I e(v\eta)\,\mathrm{d}v+O(T_0^3Y^2N^{-1}),$$

where

$$K_\chi=\frac{1}{2\pi\mathrm{i}}\int_{-T_0}^{T_0}F(t,\chi)w^{-1/2+\mathrm{i}t}\,\mathrm{d}t$$

is independent of η. Indeed,

$$S_3(\chi,\eta)=K_\chi(t(\eta)+O(1))+O(T_0^3Y^2N^{-1}) \tag{2.20}$$

by an application of Titchmarsh (1986, lemma 4.8).

We can evaluate K_χ by taking $\eta = 0$ in (2.20):

$$\begin{aligned} K_\chi(Y + O(1)) &= S_3(\chi, 0) + O(T_0^3 Y^2 N^{-1}) \\ &= S_1(\chi, 0) + O(Y\mathcal{L}^{-2A}) \\ &= \frac{\delta_\chi u(\boldsymbol{b})Y}{\mathcal{L}} + O(Y\mathcal{L}^{-2A}). \end{aligned} \tag{2.21}$$

Here we appeal to (2.6)–(2.9), (2.19) for the second equality and (1.7) for the final step. We now assemble (2.20) and (2.21) to reach (2.5) and the crucial approximation (2.4).

Applying Parseval's equality to $(1 - \Gamma)(f - f_0)g$, we see that in lieu of (2.1) we need only establish

$$\sum_{2n \in K} \left| \int_{\mathcal{M}} f_0(\alpha)g(\alpha)e(-2n\alpha)\,\mathrm{d}\alpha - \frac{u(\boldsymbol{b})u(\boldsymbol{c})}{\mathcal{L}^2}(\tfrac{2}{3}N - 2n)\mathfrak{S}(2n) \right|^2 \ll Y^3\mathcal{L}^{-A-7}. \tag{2.22}$$

Now

$$\begin{aligned} \int_{\mathcal{M}} f_0(\alpha)g(\alpha)e(-2n\alpha)\,\mathrm{d}\alpha &= \frac{u(\boldsymbol{b})}{\mathcal{L}} \sum_{q \leqslant \mathcal{L}^{2A}} \frac{\mu(q)}{\phi(q)} \sideset{}{^*}\sum_{r=1}^{q} e\left(-\frac{2nr}{q}\right) \\ &\quad \times \sum_{k,m \in I} c(m)e\left(\frac{mr}{q}\right) \int_{-1/qQ}^{1/qQ} e((k + m - 2n)\eta)\,\mathrm{d}\eta \\ &= H(n) - \int_{-1/2}^{1/2} w(\eta)e(-2n\eta)\,\mathrm{d}\eta, \end{aligned} \tag{2.23}$$

where

$$H(n) = \frac{u(\boldsymbol{b})}{\mathcal{L}} \sum_{q \leqslant \mathcal{L}^{2A}} \frac{\mu(q)}{\phi(q)} \sideset{}{^*}\sum_{r=1}^{q} e\left(-\frac{2nr}{q}\right) \sum_{m \in I} c(m)e\left(\frac{mr}{q}\right) \sum_{\substack{k \in I \\ k+m=2n}} 1,$$

$$w(\eta) = \frac{u(\boldsymbol{b})}{\mathcal{L}} \sum_{q \leqslant \mathcal{L}^{2A}} \frac{\mu(q)}{\phi(q)} \sideset{}{^*}\sum_{r=1}^{q} e\left(-\frac{2nr}{q}\right) g\left(\frac{r}{q} + \eta\right) t(\eta), \quad \left(|\eta| > \frac{1}{qQ}\right),$$

and $w(\eta) = 0$ for $|\eta| < 1/qQ$. By Parseval's equality,

$$\begin{aligned} \sum_{2n \in K} \left| \int_{-1/2}^{1/2} w(\eta)e(-2n\eta)\,\mathrm{d}\eta \right|^2 &\leqslant \int_{-1/2}^{1/2} |w(\eta)|^2\,\mathrm{d}\eta \\ &\ll \mathcal{L}^{4A} \max_{|\eta| \in (\mathcal{L}^{-2A}Q^{-1}, 1/2]} |t(\eta)|^2 \int_{-1/2}^{1/2} |g(\nu)|^2\,\mathrm{d}\nu \\ &\ll \mathcal{L}^{8A}Q^2 Y\mathcal{L}^B \ll Y^3\mathcal{L}^{-A-7}. \end{aligned} \tag{2.24}$$

Now let $I' = I \cap (2n - I)$, then I' is an interval of length $\frac{2}{3}N - 2n$ and

$$H(n) = \frac{u(\boldsymbol{b})}{\mathcal{L}} \sum_{q \leqslant \mathcal{L}^{2A}} \frac{\mu(q)}{\phi(q)} \sideset{}{^*}\sum_{r=1}^{q} e\left(-\frac{2nr}{q}\right) \sum_{m \in I'} c(m)e\left(\frac{mr}{q}\right).$$

By (1.7) for $\boldsymbol{c}$, the innermost sum is

$$\sum_{m\in I'} c(m) \sum_{b=1}^{q} e\left(\frac{b}{q}\right) \frac{1}{\phi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} \bar{\chi}(b)\chi(mr)$$
$$= \frac{1}{\phi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} \chi(r) \sum_{b=1}^{q} \bar{\chi}(b) e\left(\frac{b}{q}\right) \sum_{m\in I'} c(m)\chi(m)$$
$$= \frac{1}{\phi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} \chi(r) \sum_{b=1}^{q} \bar{\chi}(b) e\left(\frac{b}{q}\right) \left\{ \frac{\delta_\chi u(\boldsymbol{c})}{\mathcal{L}}(\tfrac{2}{3}N - 2n) + O(Y\mathcal{L}^{-5A}) \right\}$$
$$= \frac{\mu(q)}{\phi(q)} \frac{u(\boldsymbol{c})}{\mathcal{L}}(\tfrac{2}{3}N - 2n) + O(Y\mathcal{L}^{-3A}),$$

leading to

$$H(n) = \frac{u(\boldsymbol{b})u(\boldsymbol{c})}{\mathcal{L}^2}(\tfrac{2}{3}N - 2n) \sum_{q\leqslant \mathcal{L}^{2A}} \frac{\mu^2(q)}{\phi^2(q)} c_q(2n) + O(Y\mathcal{L}^{-A})$$

with

$$c_q(2n) = \sum_{r=1}^{q}{}^{*} e\left(-\frac{2nr}{q}\right).$$

Since

$$\mathfrak{S}(2n) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} c_q(2n),$$

we find that

$$\sum_{2n\in K} \left| H(n) - \frac{u(\boldsymbol{b})u(\boldsymbol{c})}{\mathcal{L}^2}\mathfrak{S}(2n)(\tfrac{2}{3}N - 2n)\right|^2$$
$$\ll Y^2 \sum_{2n\in K} \left| \sum_{q>\mathcal{L}^{2A}} \frac{\mu^2(q)}{\phi^2(q)} c_q(2n)\right|^2 + Y^3\mathcal{L}^{-2A}$$
$$\ll Y^2\mathcal{L}^{-4A+1} \sum_{j\in K} \tau^2(j) + Y^3\mathcal{L}^{-2A} \ll Y^3\mathcal{L}^{-2A} \tag{2.25}$$

(compare Baker *et al.* (1997, (2.7)–(2.9)).

Combining (2.23)–(2.25) we obtain (2.22). This completes the proof of theorem 1.2.

3. The class of sequences $\mathcal{B} \cap \mathcal{C}$

In constructing $\mathcal{B}$ we are more restricted than in Baker *et al.* (1997, § 4). (We would get the same set of sequences if we had $T' \ll T$, rather than $T' \ll T^2$, in (1.12).)

Nevertheless, we may begin with a transfer of results from Baker *et al.* (1997, §3). Suppose that

$$M(s,\chi) = \sum_{m\sim M} a_m m^{-s}\chi(m), \quad J(s,\chi) = \sum_{j\sim J} f_j j^{-s}\chi(j), \quad L(s,\chi) = \sum_{\ell\sim L} g_\ell \ell^{-s}\chi(\ell)$$

are Dirichlet polynomials; $m \sim M$ means $\frac{1}{2}M < m \leqslant M$. If $N = MJL$,

$$b'(k) = \sum_{mj\ell=k} a_m f_j g_\ell,$$

then (1.11) becomes

$$\int_{T'}^{T'+T} |(MJL)(\tfrac{1}{2} + \mathrm{i}t, \chi)| \,\mathrm{d}t \ll TYN^{-1/2}\mathcal{L}^{-A}. \tag{3.1}$$

In proving results of the form (3.1), we will need hypotheses of the shape:

$$M(\tfrac{1}{2} + \mathrm{i}t, \chi) \ll M^{1/2}\mathcal{L}^{-A} \quad \text{for all } A > 0, \tag{3.2}$$

for $T_0 \leqslant t \leqslant N$, χ a character $(\operatorname{mod} q)$, $q \leqslant \mathcal{L}^A$.

In what follows, we suppose that $\frac{4}{7} \leqslant \theta < \frac{7}{12}$, and that χ is as in (3.2).

Lemma 3.1. *Suppose M, J, L are Dirichlet polynomials whose coefficients satisfy (1.9), and L satisfies (3.2). Let $M = N^{\alpha_1}$, $J = N^{\alpha_2}$,*

$$|\alpha_1 - \alpha_2| < \tfrac{1}{7} - \epsilon, \tag{3.3}$$

$$\alpha_1 + \alpha_2 > \tfrac{60}{77} + \epsilon. \tag{3.4}$$

Then whenever (1.12) holds, we have (3.1).

Proof. In theorem 4 of Baker *et al*. (1997), take $X = N$, $T = NY^{-1}$, $q \leqslant \mathcal{L}^A$, so that $qT = N^{1-\theta'}$ with $\frac{1}{2} + \epsilon < \theta' < \frac{7}{12}$, $2\theta' - 1 \geqslant \frac{1}{7} - \frac{1}{2}\epsilon$, $\frac{1}{11}(20\theta' - 9) > \frac{17}{77} - \frac{1}{2}\epsilon$. The Dirichlet polynomials to which we apply theorem 4 are

$$M(\tfrac{1}{2} + \mathrm{i}T' + \mathrm{i}t, \chi)$$

and similarly for J, L; and we find that

$$\int_{T'}^{T'+NY^{-1}} |(MJL)(\tfrac{1}{2} + \mathrm{i}t)| \,\mathrm{d}t \ll N^{1/2}\mathcal{L}^{-A}$$

for all $A > 0$ and

$$0 \leqslant T' \leqslant T' + NY^{-1} \leqslant N$$

from (3.9) of Baker *et al*. (1997). The inequality (3.1) follows on combining this bound for $O(TN^{-1}Y)$ intervals of length NY^{-1}.

The device of division into subintervals of length NY^{-1} will be used in lemmata 3.2–3.4 without further comment. ■

Lemma 3.2. *The analogue of lemma 3.1, with (3.3), (3.4) replaced by*

$$\max(\alpha_1, \alpha_2) \leqslant 0.46 + \epsilon/2, \quad \min(\alpha_1, \alpha_2) \geqslant \tfrac{2}{7} + \epsilon, \tag{3.5}$$

$$\alpha_1 + \alpha_2 \geqslant \tfrac{36}{49} + \epsilon, \tag{3.6}$$

is valid.

Proof. Inserting the bound $qT \leqslant \mathcal{L}^{2A}N^{3/7}$ into the proof of lemma 4 of Baker *et al*. (1997), we readily obtain the required result. ■

Lemma 3.3. *The analogue of lemma 3.1, with (3.3), (3.4) replaced by*

$$\alpha_1 > \tfrac{3}{7} + \epsilon, \tag{3.7}$$

$$\alpha_2 > \tfrac{2}{7} + \epsilon, \tag{3.8}$$

$$4\alpha_1 + \alpha_2 < \tfrac{16}{7} - \epsilon, \tag{3.9}$$

is valid.

Proof. In the proof of lemma 4 of Baker *et al.* (1997), cases 2 and 4 cannot arise, because of (3.7). In case 3, the conditions needed are

$$\tfrac{1}{4}(1-\theta) + \tfrac{1}{8} + \tfrac{1}{8}(3 - 3\alpha_2) < \tfrac{1}{2} - \tfrac{1}{8}\epsilon, \quad \tfrac{1}{2}(1-\theta) + \tfrac{1}{2}\alpha_1 + \tfrac{1}{8}\alpha_2 < \tfrac{1}{2} - \tfrac{1}{8}\epsilon,$$

and these are implied by (3.8), (3.9). The proof now goes through as before. ■

Lemma 3.4. *Let L, M, J and*

$$R(s,\chi) = \sum_{r\sim R} d_r r^{-s}\chi(r) \tag{3.10}$$

satisfy (1.9), while L, J and R satisfy (3.2). Suppose further that

$$M \geqslant N^{3/7+\epsilon}, \quad J \geqslant N^{1/7+\epsilon}, \quad R^2L \geqslant N^{3/7+\epsilon}, \quad L \geqslant N^{6/35+\epsilon}.$$

Then

$$\int_{T'}^{T'+T} |MJLR(\tfrac{1}{2} + \mathrm{i}t, \chi)|\,\mathrm{d}t \ll TYN^{-1/2}\mathcal{L}^{-A}.$$

whenever (1.12) holds.

Proof. We follow the proof of lemma 3(iii) of Baker *et al.* (1997), inserting the bound $qT \leqslant \mathcal{L}^{2A}N^{3/7}$, to get the desired inequality.

We now turn to Dirichlet polynomials with special coefficients. The next lemma follows from Baker *et al.* (1997, lemmata 5, 6). ■

Lemma 3.5. *Let*

$$M(s,\chi) = \sum_{\substack{M<m\leqslant M' \\ (m,P(z))=1}} \chi(m)m^{-s}, \quad M_0(s,\chi) = \sum_{M<m\leqslant M'} \chi(m)m^{-s}, \tag{3.11}$$

where $M' \leqslant 2M$ and $z \geqslant \exp(\mathcal{L}^{9/10})$. Then $M(s,\chi)$ satisfies (3.2). If $M_0 \geqslant N^{\epsilon}$, then M_0 satisfies (3.2).

Lemma 3.6. *With M_0, R as in (3.11), (3.10), suppose that R satisfies (1.9) and*

$$R \ll YN^{-\epsilon}.$$

Whenever (1.12) holds, we have

$$\int_{T'}^{T+T'} |(M_0R)(\tfrac{1}{2} + \mathrm{i}t, \chi)|\,\mathrm{d}t \ll TYN^{-1/2}\mathcal{L}^{-A}. \tag{3.12}$$

Proof. By Cauchy's inequality and lemma 1 of Baker *et al*. (1997), the left-hand side of (3.12) is

$$\ll \mathcal{L}^B (R + qT)^{1/2} \left(\int_{T'}^{T+T'} |M_0(\tfrac{1}{2} + \mathrm{i}t, \chi)|^2 \,\mathrm{d}t \right)^{1/2}.$$

The reflection principle, as used for example in Perelli *et al*. (1984, (21)) yields

$$M_0(\tfrac{1}{2} + \mathrm{i}t, \chi) = J(\tfrac{1}{2} - \mathrm{i}t, \chi') + O(1),$$

for $t \in [T', T + T']$, where the Dirichlet polynomial J has coefficients of modulus 1, χ' is χ or $\bar{\chi}$, and J has length $\ll (T' + T)^{1/2} \ll T$. Hence

$$\int_{T'}^{T+T'} |M_0(\tfrac{1}{2} + \mathrm{i}t, \chi)|^2 \,\mathrm{d}t \ll qT$$

by another application of Baker *et al*. (1997, lemma 1). The left-hand side of (3.12) is

$$\ll \mathcal{L}^B (YN^{-\epsilon})^{1/2} T^{1/2} \ll TYN^{-1/2}\mathcal{L}^{-A},$$

since $T \geqslant NY^{-1}$.

Let

$$\psi(n, z) = \begin{cases} 1 & \text{if } (n, P(z)) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geqslant 1$, $z \geqslant 2$, and write

$$w = \exp(\mathcal{L}^{9/10}), \quad z_0 = N^{1/7 - 2\epsilon}.$$

■

Lemma 3.7. *Let* $\boldsymbol{b} = (b(k))_{k \in I}$, *where*

$$b(k) = \sum_{\substack{m \sim M \\ m\ell = k}} a_m \psi(\ell, w)$$

where a_m *satisfies (1.9),* $a_m = 0$ *for* $(m, P(w)) > 1$, *and*

$$M \ll N^{1/2}.$$

Then $\boldsymbol{b} \in \mathcal{B} \cap \mathcal{C}$.

Proof. Let

$$b'(k) = \sum_{\substack{m \sim M \\ ndm = k}} a_m \sum_{d | P(w),\, d \leqslant N^{\epsilon/2}} \mu(d);$$

then, just as in the proof of Baker *et al*. (1997, (4.12)), we obtain (1.10). Lemma 3.7 now follows readily from lemma 3.6 and the definition of a type-(I) sum. ■

Lemma 3.8. *Let*

$$b(k) = \sum_{\substack{m \sim M \\ m\ell = k}} a_m \psi(\ell, z),$$

where $M \leqslant N^{1/2}$, a_m *satisfies (1.9),* $a_m = 0$ *for* $(m, P(w)) > 1$, *and*

$$w \leqslant z \leqslant z_0.$$

Then $\boldsymbol{b} \in \mathcal{B} \cap \mathcal{C}$.

Proof. Let $q_1, q_2, \ldots$ be prime variables. We apply Buchstab's identity

$$\psi(j,z) = \psi(j,w) - \sum_{\substack{ph=j \\ w \leqslant p < z}} \psi(h,p)$$

(Baker *et al*. 1997 (4.14)) to obtain

$$b(k) = \sum_{\substack{m\ell=k \\ m\sim M}} a_m \psi(\ell, w) - \sum_{\substack{mq_1h=k \\ m\sim M,\, w\leqslant q_1<z}} a_m \psi(h, p_1) = b_0'(k) - b_1'(k),$$

say. Let $b_1(k)$ be the subsum of $b_1'(k)$ defined by the extra condition

$$mq_1^{1/2} < N^{3/7+\epsilon}$$

and $b_1''(k)$ the complementary subsum, so that

$$b(k) = b_0'(k) - b_1(k) - b_1''(k).$$

We now apply Buchstab's identity to $b_1(k)$. In this fashion we may obtain successive decompositions

$$b_j(k) = b_j'(k) - b_{j+1}(k) - b_{j+1}''(k)$$

with

$$b_j(k) = \sum_{mq_1\ldots q_j h=k} a_m \psi(h, p_j),$$

the summation being restricted by

$$m \sim M, w \leqslant q_j < \cdots < q_1 < z, \tag{3.13$_j$}$$

$$mq_1 \ldots q_{j-1} q_j^{1/2} < N^{3/7+\epsilon}, \tag{3.14$_j$}$$

$b_j'(k)$ defined as $b_j(k)$ with $\psi(h,w)$ in place of $\psi(h,p_j)$, and $b_{j+1}''(k)$ defined in the same way as $b_{j+1}(k)$ with (3.14$_{j+1}$) replaced by: (3.14$_j$) *and*

$$mq_1 \ldots q_j q_{j+1}^{1/2} \geqslant N^{3/7+\epsilon}. \tag{3.15$_{j+1}$}$$

After less than $\mathcal{L}$ steps, $b_j(k)$ is empty and decomposition ceases. From (3.13$_j$), (3.14$_j$),

$$mq_1 \ldots q_{j-1} q_j \ll N^{1/2} \tag{3.16}$$

for the terms of $b_j'(k)$, and we may apply lemma 3.7 to show that $\boldsymbol{b}_j'$ is in $\mathcal{B} \cup \mathcal{C}$.

We now consider the sequence $\boldsymbol{b}_{j+1}'' = (b_{j+1}''(k))_{k\in I}$, where $j \geqslant 0$. We shall deduce that $\boldsymbol{b}_{j+1}'' \in \mathcal{B}$ from lemma 3.1. The interdependence of the variables arising from the factor $\psi(h,p_j)$ is removed by the procedure described on pp. 27, 28 of Baker *et al*. (1997). The Dirichlet polynomials that we use have

$$M = N^{\alpha_1}, \quad J = N^{\alpha_2}, \quad L \leqslant N^{1/7-\epsilon}.$$

Here M corresponds to $mq_1 \ldots q_j$, L to q_{j+1}, and J to h. In view of lemma 3.7, the requirements (1.9), and (3.2) for L, are met. Since

$$\alpha_1 + \alpha_2 \geqslant \tfrac{6}{7} + \epsilon, \tag{3.17}$$

the remaining condition that we need to verify is

$$-\tfrac{1}{7} + \epsilon \leqslant \alpha_1 - \alpha_2 \leqslant \tfrac{1}{7} - \epsilon. \tag{3.18}$$

The left-hand inequality comes directly from (3.15_{j+1}). Moreover,

$$\begin{aligned} \alpha_1 - \alpha_2 &= 2\alpha_1 - (\alpha_1 + \alpha_2) \\ &\leqslant 1 - (\tfrac{6}{7} + \epsilon) = \tfrac{1}{7} - \epsilon \end{aligned}$$

from (3.16), (3.17). It follows that $\boldsymbol{b}''_{j+1} \in \mathcal{B}$.

To see that $\boldsymbol{b}''_{j+1} \in \mathcal{C}$, we simply observe that if ℓ is the least integer with $mq_1 \dots q_\ell \geqslant N^{3/7+\epsilon}$, then $\ell = j$ or $j+1$ from (3.14_j), (3.15_{j+1}). Consequently,

$$N^{3/7+\epsilon} \leqslant mq_1 \dots q_\ell < N^{3/7+\epsilon} z \leqslant N^{4/7-\epsilon},$$

if $\ell > 0$, while if $\ell = 0$, then

$$N^{3/7+\epsilon} \leqslant m < 2N^{1/2}.$$

This completes the proof of lemma 3.8. ■

We now give a sequence $A_0(k)$ in $\mathcal{B} \cap \mathcal{C}$ for which

$$\rho(k) \geqslant A_0(k). \tag{3.19}$$

By Buchstab's identity,

$$\begin{aligned} \rho(k) &= \psi(k, z_0) - \sum_{\substack{z_0 \leqslant p_1 < N^{1/2} \\ p_1 p_2 = k}} \psi(n_2, z_0) + \sum_{\substack{p_1 p_2 n_3 = k \\ z_0 \leqslant p_2 < p_1 < N^{1/2}}} \psi(n_3, p_2) \\ &= S_1(k) - S_2(k) + S_3(k), \quad \text{say.} \end{aligned}$$

We do not decompose further those parts of $S_3(k)$ for which either

$$p_1 p_2^2 > N^{4/7-\epsilon} \tag{3.20}$$

or

$$p_1 p_2 < N^{5/14+\epsilon} \quad \text{and} \quad p_1 p_2^2 > N^{1/2}. \tag{3.21}$$

Writing $\sum'$ for a sum in which neither (3.20) nor (3.21) holds, we decompose twice more to obtain

$$\begin{aligned} \sideset{}{'}\sum_{\substack{p_1 p_2 p_3 = k \\ z_0 \leqslant p_2 < p_1 < N^{1/2}}} \psi(n_3, p_2) &= \sideset{}{'}\sum_{\substack{p_1 p_2 n_3 = k \\ z_0 \leqslant p_2 < p_1 < N^{1/2}}} \psi(n_3, z_0) - \sideset{}{'}\sum_{\substack{z_0 \leqslant p_3 < p_2 < p_1 < N^{1/2} \\ p_1 p_2 p_3 n_4 = k}} \psi(n_4, z_0) \\ &\quad + \sideset{}{'}\sum_{\substack{z_0 \leqslant p_4 < p_3 < p_2 < p_1 \\ p_1 p_2 p_3 p_4 n_5 = k}} \psi(n_3, p_4) \\ &= S_4(k) - S_5(k) + S_6(k), \end{aligned}$$

say. We can now 'recover' some of the terms of $S_3(k)$. Suppose that p_1, p_2 lie in a region satisfying (3.20) or (3.21) for which some subproduct of the variables p_1, p_2, n_3 lies in $[N^{3/7+\epsilon}, N^{4/7-\epsilon}]$ *and* some arrangement of variables permits application of lemma 3.1, 3.2 or 3.3. Let us denote by $S_{3,1}(k)$ this portion of the sum $S_3(k)$; then $(S_{3,1}(k)) \in \mathcal{B} \cap \mathcal{C}$.

Similarly, suppose that p_1, p_2, p_3, p_4 lie in a part of the domain of summation of $S_6(k)$ for which some subproduct of the variables p_1, p_2, p_3, p_4, n_5 lies in $[N^{3/7+\epsilon}, N^{4/7-\epsilon}]$ *and* some arrangement of the variables permits application of lemma 3.1, 3.2, 3.3 or 3.4. Let us denote by $S_{6,1}(k)$ this portion of the sum $S_6(k)$; then $(S_{6,1}(k)) \in \mathcal{B} \cap \mathcal{C}$. We now have (3.19) with

$$A_0(k) = S_1(k) - S_2(k) + S_{3,1}(k) + S_4(k) - S_5(k) + S_{6,1}(k).$$

Lemma 3.9. *The sequence* $\boldsymbol{A}_0$ *is in* $\mathcal{B} \cap \mathcal{C}$.

Proof. In view of the above discussion and lemma 3.8 it suffices to prove that

$$(S_4(k)) \in \mathcal{B} \cap \mathcal{C}, \tag{3.22}$$

$$(S_5(k)) \in \mathcal{B} \cap \mathcal{C}. \tag{3.23}$$

In $S_4(k)$ we have $p_1p_2 = p_1p_2^2p_2^{-1} \leqslant N^{4/7-\epsilon}p_2^{-1}$, and so

$$p_1p_2 < N^{3/7+\epsilon}.$$

Thus (3.22) is a consequence of lemma 3.7.

We examine first the part of $S_5(k)$ satisfying

$$p_1p_2p_3^{1/2} \geqslant N^{3/7+\epsilon}. \tag{3.24}$$

Since

$$p_1p_2p_3 \leqslant p_1p_2^2 \leqslant N^{4/7-\epsilon},$$

this part of $S_5(k)$ is in $\mathcal{C}$. Moreover,

$$\frac{p_1p_2}{k/(p_1p_2p_3)} = \frac{p_1^2p_2^2p_3}{k} \geqslant N^{-1/7+\epsilon},$$

$$\frac{p_1p_2}{k/(p_1p_2p_3)} \leqslant \frac{p_1^2p_2^3}{k} \leqslant N^{1/7-\epsilon}.$$

By lemma 3.1, this part of $S_5(k)$ is in $\mathcal{B} \cap \mathcal{C}$.

Turning to the part of $S_5(k)$ for which (3.24) is violated, we note that

$$p_1p_2 < N^{3/7+\epsilon}p_3^{-1/2} < N^{5/14+\epsilon}.$$

Since (3.21) is violated,

$$m = p_1p_2p_3 \leqslant p_1p_2^2 \leqslant N^{1/2}.$$

We now deduce from lemma 3.8 that this part of $S_5(k)$ is in $\mathcal{B} \cap \mathcal{C}$.

For the sequence $A_1(k)$, we begin by noting that

$$\rho(k) = \psi(k, z_0) - \sum_{\substack{z_0 \leqslant p_1 < N^{1/4} \\ p_1 n_2 = k}} \psi(n_2, z_0) - \sum_{N^{1/4} \leqslant p_1 < N^{1/2}} \psi(n_2, p_1) + \sum_{\substack{p_1p_2n_3 = k \\ z_0 \leqslant p_2 < p_1 < N^{1/4}}} \psi(n_3, z_0) - \sum_{\substack{p_1p_2p_3n_4 = k \\ z_0 \leqslant p_3 < p_2 < p_1 < N^{1/4}}} \psi(n_4, p_3);$$

that is, we decompose twice more the part of $\psi(n_2, p_1)$ having $p_1 < N^{1/4}$. Let us write this as

$$\rho(k) = T_1(k) - T_2(k) - T_3(k) + T_4(k) - T_5(k),$$

say. Let $T_{5,1}(k)$ denote the part of $T_5(k)$ for which some subproduct of the variables lies in $[N^{3/7+\epsilon}, N^{4/7-\epsilon}]$ *and* some arrangement of variables permits application of lemma 3.1, 3.2, 3.3 or 3.4. Then

$$\rho(k) \leqslant A_1(k),$$

where

$$A_1(k) = T_1(k) - T_2(k) + T_4(k) - T_{5,1}(k).$$

It is clear from lemma 3.8 and the definition of $T_{5,1}(k)$ that $\boldsymbol{A}_1 \in \mathcal{B} \cap \mathcal{C}$. The requirement that $\boldsymbol{A}_0, \boldsymbol{A}_1$ satisfy (1.7) can be established in the same way as in Baker *et al*. (1997); namely, via lemma 11 of Baker *et al*. (1997) together with the Siegel–Walfisz theorem.

As for the constants $u(\boldsymbol{A}_0)$ and $u(\boldsymbol{A}_1)$, a computer calculation yields

$$u(\boldsymbol{A}_0) > \tfrac{1}{4}.$$

With a little thought we see that $u(\boldsymbol{A}_1) = 4w(4) + c$, where w is Buchstab's function and c is a three-dimensional integral corresponding to $T_5 - T_{5,1}$ (cf. Baker *et al*. 1997, p. 53). Since $c < 0.04$ by a computer calculation, we obtain

$$u(\boldsymbol{A}_1) < 2.32.$$

In § 4 of Baker *et al*. (1997), we find sequences $B_0(m)$ and $B_1(m) (m \in I)$ in $\mathcal{B}_0$ which satisfy

$$B_0(m) \leqslant \rho(m) \leqslant B_1(m), \tag{3.25}$$

$$0.99 < u(\boldsymbol{B}_0) < 1 < u(\boldsymbol{B}_1) < 1.01. \tag{3.26}$$

Since

$$u(\boldsymbol{A}_0)u(\boldsymbol{B}_1) + u(\boldsymbol{A}_1)(u(\boldsymbol{B}_0) - u(\boldsymbol{B}_1)) > \tfrac{1}{4} - 2.32 \times 0.02 > \tfrac{1}{5},$$

(1.6) holds. This completes the proof of theorem 1.1. ■

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